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Analytic Solutions to Several Optimum Orbit Transfer Problems*

H. Munick, R. McGill and G. E. Taylor†

bstract

The analytic solution is given for the *absolute minimum* aracteristic-velocity path from a point on an elliptical bit to a nonintersecting circular orbit. The orbits are condered coplanar and two impulses are used. Existence proofs e given followed by proofs that the absolute minimum tisfies both necessary and sufficient conditions.

More generally the complete analytic solution to the rminal-to-terminal problem as first formulated by Vargo][1] is given.

An interesting consequence of the analytic solution is the roof that in optimal transfer the impulses are not in general pplied tangentially. Impulses are applied tangentially only terminals having zero radial velocity.

Each point on an elliptical orbit corresponds to an arrival oint of an optimum path from a lower energy circular orbit. he Hohmann path arriving at apogee is rigorously shown to e the optimum of the infinity of optimal transfer paths, for a ide class of orbits.

ymbols

= normal component of velocity
= radial component of velocity
= nondimensional normal component of velocity just after first impulse
= nondimensional radial component of velocity just after first impulse
= initial radial distance
= final radial distance
= distance ratio = α/β
= nondimensional characteristic velocity
= characteristic velocity of transfer
= gravitational constant
= nondimensional normal component of arrival velocity
= nondimensional radial component of arrival velocity
bscripts as defined in text

* Presented at the 11th International Astronautical ongress, Stockholm, Sweden, August 14–20, 1960.

† Grumman Aircraft Engineering Corporation, Bethpage, ew York.

[1] Numbers in brackets indicate references at end of paper.

Introduction

The problem solved in this paper is that of finding the analytic expressions for the minimum characteristic-velocity path (hereafter called the optimum path) for a class of orbital transfers. The optimum path solutions are derived for transfer between two terminals, and between a terminal and a "point." For the purposes of this paper we define a terminal as a locus specified by a radial distance and a velocity vector, i.e., a position on an ellipse with the line of apsides unspecified. A "point" is defined by a radius vector and a velocity vector. A "point" then, is a specific position on an ellipse whose line of apsides *is* specified. The class of orbits considered are those for which the major apsis of one terminal is less than the minor apsis of the other terminal. This is tantamount to saying that a minimum of two impulses is required to accomplish the transfer.

The formulation given herein is that of Ref. [2] which was pioneered by Lawden.

The assumptions of the paper are:

1) Inverse square force field
2) Two-body equations
3) All orbits are coplanar
4) Impulsive thrusting
5) Two impulses are used.

Definition of Problem

Consider the problem of optimum transfer of a space vehicle between two terminals. Let (u_0, v_0) be the velocity components of the first terminal at a distance α from the attractive center. Let (u_F, v_F) be the velocity components of the second terminal at a distance β from the attractive center. At the first terminal an impulse is applied resulting in new velocity components (u_1, v_1). The space vehicle then goes into a transfer orbit, arriving at the second terminal with velocity components (u_2, v_2). Upon arrival at the second terminal a second impulse is applied adjusting the arrival velocity to the desired terminal velocity (u_F, v_F). For such a maneuver the characteristic-velocity is given by

$$\lambda^* = \sqrt{(u_1 - u_0)^2 + (v_1 - v_0)^2} + \sqrt{(u_2 - u_F)^2 + (v_2 - v_F)^2}. \quad (1)$$

From conservation of angular momentum we have

$$u_2 = \frac{\alpha}{\beta} u_1, \quad (2)$$

and from the conservation of total energy

$$v_2^2 = \left[1 - \frac{\alpha^2}{\beta^2}\right] u_1^2 + v_1^2 + 2\mu\left[\frac{1}{\beta} - \frac{1}{\alpha}\right]. \quad (3)$$

Introduce dimensionless parameters x, y by:

$$u_1 = x\sqrt{\frac{\mu}{\alpha}}, \qquad v_1 = y\sqrt{\frac{\mu}{\alpha}}, \quad (4)$$

and in all other cases:

$$u_i = x_i\sqrt{\frac{\mu}{\alpha}}, \qquad v_i = y_i\sqrt{\frac{\mu}{\alpha}}. \quad (5)$$

Let the distance ratio be denoted by

$$r = \frac{\alpha}{\beta}. \quad (6)$$

Using (2), (4), (5), and (6), the characteristic-velocity becomes

$$\lambda^* = \sqrt{\frac{\mu}{\alpha}}\,[\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2}]. \quad (7)$$

And Eq. (3) becomes

$$y_2^2 = (1 - r^2)x^2 + y^2 + 2(r - 1). \quad (8)$$

From (7) and (8) we get

$$\lambda^* \Big/ \sqrt{\frac{\mu}{\alpha}} = \sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2}. \quad (9)$$

The sign of the radical $\sqrt{(1 - r^2)x^2 + y^2 + 2(r - 1)}$ as obtained from (8), must be taken the same as y_f. This is dictated by the fact that we are minimizing characteristic-velocity.

Let $\lambda(x, y)$ denote the nondimensional characteristic-velocity as given by the right-hand side of (9). From (8) we observe the constraint relationship that

$$(1 - r^2)x^2 + y^2 + 2(r - 1) \geqq 0. \quad (10)$$

If (10) is violated an imaginary radial velocity results. The physical interpretation of this is that the vehicle cannot achieve a distance β from the focus; hence a transfer in this case is impossible.

From (8) and (9), we can now define the mathematical problem of interest to be that of minimizing

$$\lambda(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2} \quad (11)$$

subject to

$$(1 - r^2)x^2 + y^2 + 2(r - 1) \geqq 0, \quad (12)$$

with

$$0 < r < 1. \quad (13)$$

Solution to Problem

In [2] it is rigorously proved that the functio[n] $\lambda(x, y)$ given by (11), will assume a relative minimu[m] at an interior point bounded by the closed curv[e] $y_2 = 0$ and $y_2 = y_F$. Since $\lambda(x, y)$ is a differentiab[le] function in such a region, a necessary condition for th[e] solution (x, y) of the minimum problem is given by:

$$\frac{\partial\lambda}{\partial x} = \frac{\partial\lambda}{\partial y} = 0. \quad (14)$$

It is also shown in [2] that using the two necessar[y] conditions given by (14) we establish for $y_2 \neq 0$

$$[(1 - r^2)x_F^2 + 2(r - 1)]K^2 - [2y_F\{(1 - r^2)rx_Fx + 2(r - 1)\}]K + [(1 - r^2)r^2y_F^2x^2 + 2(r - 1)y_F^2] = 0 \quad (15)$$

where

$$K = \sqrt{(1 - r^2)x^2 + y^2 + 2(r - 1)} = y_2.$$

Introducing the transformation $y_2' = (y_2/y_F)$ in[to] (15), and simplifying we get

$$[(1 + r)x_F^2 - 2]y_2'^2 - 2y_2'[(1 + r)rx_Fx - 2] + [(1 + r)r^2x^2 - 2] = 0. \quad (16)$$

Re-arranging terms

$$[(1 + r)r^2]x^2 - 2x[(1 + r)rx_Fy_2'] + [(1 + r)x_F^2y_2'^2 - 2(y_2' - 1)^2] = 0. \quad (17)$$

Dividing through by $(1 + r)$ we get

$$r^2x^2 - 2rx_F\,y_2'x + \left[x_F\,y_2' + (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right] \times \left[x_F\,y_2' - (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right] = 0. \quad (18)$$

Factoring (18) yields

$$\left[rx - \left\{x_F\,y_2' + (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right\}\right] \times \left[rx - \left\{x_F\,y_2' - (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right\}\right] = 0. \quad (19)$$

From (19) we conclude that the minimum f[or] $\lambda(x, y)$ must satisfy one of two possibilities:

$$rx = x_F\,y_2' + (y_2' - 1)\sqrt{\frac{2}{1 + r}}, \quad (20)$$

or

$$rx = x_F\,y_2' - (y_2' - 1)\sqrt{\frac{2}{1 + r}}. \quad (21)$$

Combining terms in (20), and using $y_2' = (y_2/y_F)$

$$rx + \sqrt{\frac{2}{1 + r}} = \frac{y_2}{y_F}\left(x_F + \sqrt{\frac{2}{1 + r}}\right). \quad (22)$$

Further simplification of (22) yields

$$\frac{y_2 - y_F}{rx - x_F} = \frac{y_F}{x_F + \sqrt{\dfrac{2}{1+r}}} \qquad (23)$$

sing (23) and the expressions for the derivatives $\partial\lambda/\partial x)$, $(\partial\lambda/\partial y)$ given in [2] we conclude

$$\frac{y - y_0}{x - x_0} = \frac{y_0}{x_0 + r\sqrt{\dfrac{2}{1+r}}}. \qquad (24)$$

Eq. (24) gives y as a linear function of x, and setting is linear function of x into (23) yields a quadratic uation in x whose coefficients depend on the given ata. We then solve analytically and determine two kact values of x.

Using the second possibility given by (21) and proeeding as above we obtain

$$\frac{y_2 - y_F}{rx - x_F} = \frac{y_F}{x_F - \sqrt{\dfrac{2}{1+r}}} \qquad (25)$$

$$\frac{y - y_0}{x - x_0} = \frac{y_0}{x_0 - r\sqrt{\dfrac{2}{1+r}}}. \qquad (26)$$

Similarly the pair of Eqs. (25), (26) can be solved alytically yielding two more exact values of x. Using 24), (26) the corresponding y values are determined.

We have now determined analytically four solutions $, y)$ to the minimum problem. Using these four lues of (x, y) there correspond four values of λ. The nallest of these four values of λ is the *absolute minimum* r λ. This then gives the complete solution to the rminal-to-terminal problem first formulated by argo [1]. It will be shown subsequently that under rtain orbital conditions three of the four values of λ n be rejected analytically. We next study geometrilly the optimum transfer path given by the above alytical expressions.

Replacing rx by x_2 in (23), we have the pair of equaon from (23), (24)

$$\frac{y_2 - y_F}{x_2 - x_F} = \frac{y_F}{x_F + \sqrt{\dfrac{2}{1+r}}}$$

$$\frac{y - y_0}{x - x_0} = \frac{y_0}{x_0 + r\sqrt{\dfrac{2}{1+r}}}. \qquad (27)$$

Eqs. (27) tells us that only in the case where the st or second terminals have radial velocities apoaching zero will the transfer orbit leave or arrive ngentially. In all other cases the transfer orbit does t leave the first terminal tangentially nor arrive at e second terminal tangentially. In the interesting se of transfer between a circular orbit and a point on nonintersecting elliptic orbit, the transfer path is ngential to the circular orbit, but is not tangent to the iptical orbit (see Figure 1).

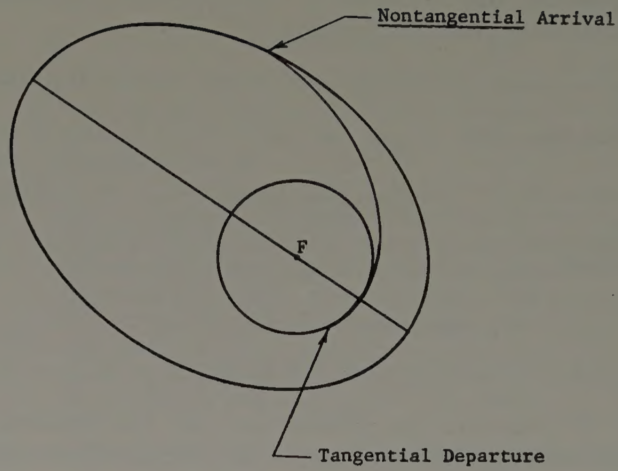

Fig. 1. Optimum Circle-to-Ellipse Transfer

Eqs. (25), (26) give the only remaining possibility, and the same tangency conditions hold for this case. We next study the case of transfer between an elliptic and a circular orbit.

Ellipse—Circle Transfer

For the case where the second terminal corresponds to a circular orbit we have

$$u_F = \sqrt{\frac{\mu}{\beta}} = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{\alpha}{\beta}} = \sqrt{\frac{\mu}{\alpha}}\, r^{1/2}$$

$$v_F = 0.$$

Nondimensionalizing we get

$$x_F = r^{1/2}, \qquad y_F = 0.$$

Since (15) does not hold for $y_2 = 0$ we first consider transfer to an almost circular orbit

$$x_F = r^{1/2}, \qquad y_F = \epsilon$$

where ϵ is an arbitrarily small radial velocity. In this case the annular region wherein the solution lies is bounded by two ellipses arbitrarily close to one another. Also the second possibility given by (25), (26) can be eliminated for positive y_F. It is shown in [2] that y_2 is less than y_F, and rx is less than x_F. That is, the transfer orbit is itself an ellipse. Therefore the left-hand side of (25) is strictly positive. Since r must lie between zero and one we conclude

$$x_F - \sqrt{\frac{2}{1+r}} = r^{1/2} - \sqrt{\frac{2}{1+r}} < 0.$$

Therefore (25) is physically unrealizable. Eq. (23) for arbitrarily small radial velocity y_F becomes

$$\frac{y_2 - \epsilon}{rx - r^{1/2}} = \frac{\epsilon}{r^{1/2} - \sqrt{\dfrac{2}{1+r}}}. \qquad (28)$$

Since ϵ can be chosen arbitrarily small we conclude from (28) that in transfer to a circular orbit y_2 becomes arbitrarily small. We then say

$$y_2^2 = (1 - r^2)x^2 + y^2 + 2(r - 1) = 0 \qquad (29)$$

and from (24)

$$\frac{y - y_0}{x - x_0} = \tan \varphi_0 \qquad (30)$$

where

$$\tan \varphi_0 = \frac{y_0}{x_0 + r\sqrt{\dfrac{2}{1 + r}}}. \qquad (31)$$

To determine the possible roots we first introduce the following

Let $A = \tan \varphi_0$

$$B = y_0 - x_0 \tan \varphi_0 = r\sqrt{\frac{2}{1 + r}} \tan \varphi_0 . \qquad (32)$$

Using these values of A, B we get from (29)

$$(1 - r^2 + A^2)x^2 + 2ABx + [B^2 - 2(1 - r)] = 0. \qquad (33)$$

Solving for x

$$x = \frac{-AB}{1 - r^2 + A^2} \pm \frac{\{A^2B^2 - (1 - r^2 + A^2)[B^2 - 2(1 - r)]\}^{\frac{1}{2}}}{1 - r^2 + A^2}. \qquad (34)$$

From (32), (34) and some simplification we get

$$x = \frac{\tan^2 \varphi_0 r\sqrt{\dfrac{2}{1 + r}} \pm (1 - r)\sqrt{2(1 + r)} \sec \varphi_0}{r^2 - \sec^2 \varphi_0}. \qquad (35)$$

Since the denominator in (35) is strictly negative we choose the negative possibility for the numerator to insure that x is given as a positive quantity. That x must be positive is determined by the fact that in this case x_0 and x_F have the same sign (both orbital rotations taken in same sense). For the retrograde case x_0 and x_F would have opposite signs and x could be negative. This gives

$$x = \frac{\tan^2 \varphi_0 r\sqrt{\dfrac{2}{1 + r}} - (1 - r)\sqrt{2(1 + r)} \sec \varphi_0}{r^2 - \sec^2 \varphi_0}. \qquad (36)$$

In general the nondimensional characteristic-velocity can be written as

$$\lambda = (x - x_0)\sqrt{1 + \left(\frac{y - y_0}{x - x_0}\right)^2} + (x_F - x_2)\sqrt{1 + \left(\frac{y_2 - y_F}{x_2 - x_F}\right)^2}. \qquad (37)$$

For the ellipse to circle case using (23), (24), (36) and (37) the absolute minimum for λ is given by

$$\lambda_{\substack{\text{absolute}\\ \text{minimum}}} = r^{1/2} + \sqrt{\frac{2}{1 + r}} - \sqrt{\left(x_0 + r\sqrt{\frac{2}{1 + r}}\right)^2 + y_0^2}. \qquad (3$$

For the special case of $x_0 = 0$, i.e., the initial veloci all radial, this result reduces to that given by Lawd [3]. Also for the case wherein $y_0 = 0$, circle-to-circ Eq. (38) reduces to the Hohmann result which hereby clearly shown to be an absolute minimum.

An analogous result may be obtained for the case circle-to-ellipse transfer.

Optimum Apogee Arrival Paths

In [2] it is demonstrated that for optimum circle-t ellipse transfer the first impulse must be appli tangentially. For this case the nondimensional chara teristic-velocity is found to be

$$\lambda = x - 1 + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2}. \qquad (3$$

From (39)

$$\lambda \geqq x - 1 + x_F - rx, \qquad (4$$

the nondimensional velocity x_F, in terms of the orbit elements, is given by

$$x_F = r\sqrt{\frac{1 + e_F}{r_p}} \qquad (4$$

where: e_F = eccentricity of elliptic orbit

r_p = distance ratio at perigee.

Since departure from the circular orbit is tangenti ($y = 0$) it follows that

$$x \geqq \sqrt{\frac{2}{1 + r}}. \qquad (4$$

This may be deduced from a study of the annul region containing the solution (see Ref. [2]). Combini (39), (40) and (41):

$$\lambda \geqq \sqrt{\frac{2}{1 + r}}(1 - r) + r\sqrt{\frac{1 + e_F}{r_p}} - 1. \qquad (4$$

The least value of the derivative with respect to r the right-hand side of (43) is

$$-\frac{3}{2}\sqrt{2} + \frac{1}{r_p}.$$

If the least value of this derivative is positive then values of the derivative are positive. Therefore t right-hand side of (43) is monotonically increasing i provided

$$r_p \leqq \frac{1}{4.5}. \qquad (4$$

t is shown in [2] that at apogee or perigee the equality n (43) holds. For the class of orbits given by (44) the ight-hand side of (43) assumes its minimum value at he smallest r, i.e., at apogee. Since from (43) it is mpossible for λ to assume a smaller value than the ninimum of the right-hand side it follows that λ assumes its least value at the smallest value of r, which ccurs at apogee.

Starting from the circular orbit, the methods given n this paper give the optimum path to any given point n the elliptical orbit. Therefore to every point on the lliptical orbit there corresponds an optimum transfer ath. Of the infinity of optimal paths there is an ptimum transfer path. It is rigorously demonstrated bove that for the class of orbits defined by (44) the Hohmann path arriving at apogee is such a least fuel path.

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Orbital Patterns for Satellite Systems

Louis G. Vargo*

Abstract

Groupings of satellites are considered as a means of performing the equipment-carrying function in several types of advanced systems. These groupings and their associated orbital arrangements or patterns are analyzed on the basis of their ability to continuously cover a large part of the Earth's surface. Optimal patterns are determined which minimize the total number of satellites required. Some operational features of these patterns are noted with the result that no fundamental technological barriers appear to exist in their realization. It is also concluded that determinate patterns as opposed to those involving probabilistic coverage are preferable.

Introduction

Several types of proposed space systems demand a continuous or nearly-continuous capability to gather information over large areas of the Earth's surface and, in some cases, transfer this information to other widely separated locations. Communication systems using line-of-sight transmission are typical. These systems have been investigated by Haviland[1] and Vilbig[2] with some treatment of the orbital problems encountered when terrestrial satellites are employed as equipment carriers. The central purpose of the present paper is to examine the orbital aspects of a broad class of systems which have the above-mentioned coverage requirements. This class may also include navigation, meterological, and military reconnaissance, surveillance and bombing objectives.

Instantaneous Coverage Areas

In general, the geographic area from which a satellite can obtain data (or, conversely, the locations which can receive information from the satellite) at any instant of time is a function of the position and velocity of the satellite relative to the Earth. For most cases, this area is quite insensitive to satellite speed and direction because of the great relative speed of the band of electromagnetic radiation used for sensing and transmission. Systems employing some type of scan device require a closer determination of the velocity-area dependence. Scan rates of the order of one cycle/sec. or greater represent current practice. This value and the greatest angular speed associated with a low-altitude circular orbit imply a displacement of the sub-satellite point of about five nautical miles per cycle. We will show that this displacement is at least two orders of magnitude down from the linear dimensions of a typical area seen by the equipment during any one cycle. On this basis one ma ignore the velocity effect in the preliminary desig phase of the orbital system and associate an area of th Earth's surface with only the instantaneous position c the satellite. This area will be referred to as the instan taneous coverage area (ICA). It contains all geographi locations at which the system purpose is accomplishe by a single satellite at one instant.

Many ICA's exhibit radial symmetry about the sub satellite point. This may result from the use of fre rotational scanning for purposes of injecting a constan angular momentum vector into the attitude contro dynamics of the vehicle. The dynamical analysis is thu simplified when compared with that required for sca techniques involving reciprocative motion. For a give scan rate, peak component accelerations and problem related to bearing design are alleviated. The rotationa axis may then be directed along the local vertical anc if slant range or incidence angle limitations are imposec these conditions can be met uniformly in all azimutha directions. Fig. 1 shows the geometric situation alon with definitions of the pertinent quantities.

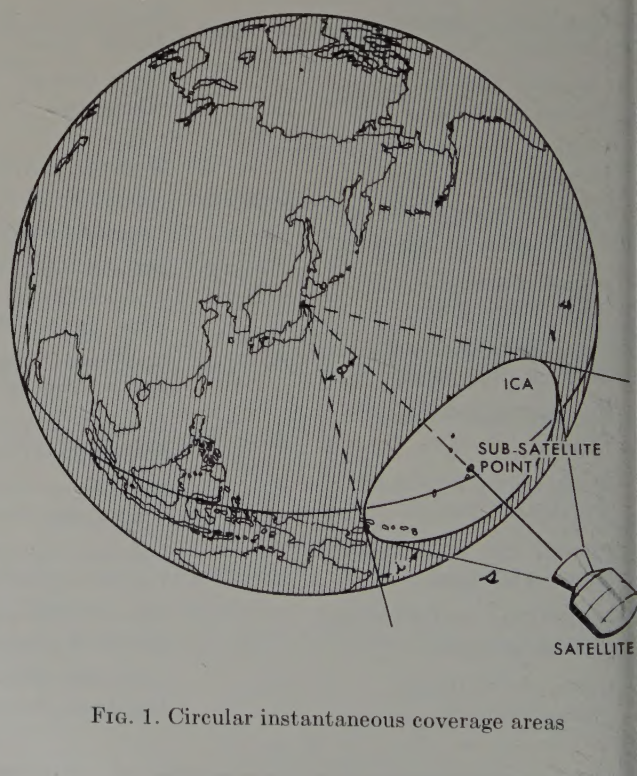

Fig. 1. Circular instantaneous coverage areas

* Research Engineer, Astrodynamics Department, Aeronutronic, A division of Ford Motor Company, Newport Beach, California.

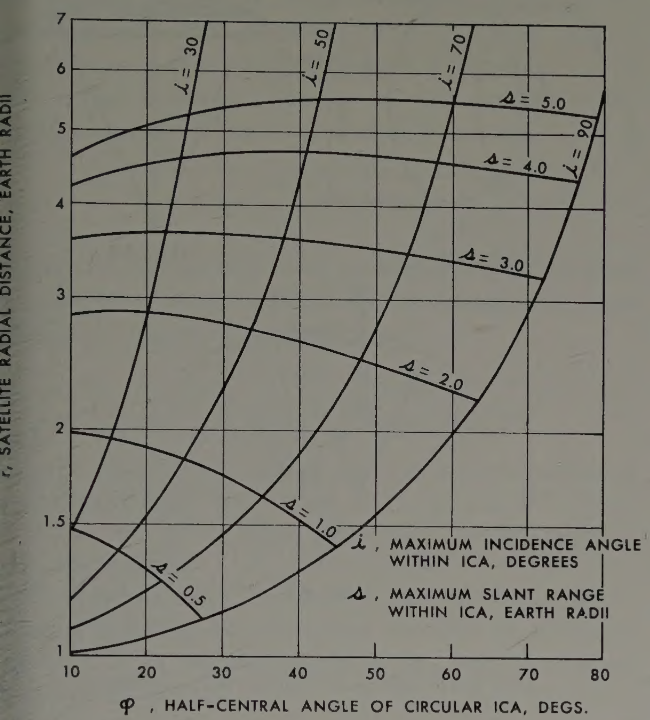

Fig. 2. Correspondence between circular ICA size and satellite distance for various incidence angle and slant range limits.

The spherical surface of the Earth implies circular ICA's for the above type of satellite equipment. This form of ICA is shared by systems employing fixed antennas which are essentially omnidirectional. Line-of-sight range and incidence angle restrictions arise on circles about the sub-satellite points. If these restrictions are not present, the ICA is bounded by the local horizon and, again, is circular. The curves of Fig. 2 are useful in determining trade-offs between equipment performance factors, satellite altitude and the ICA size. We plot the half-central angle φ vs. radial distance r for various maximum incidence angles with maximum slant range curves superimposed. For a given type of equipment, the orbital patterns for some overall coverage requirement evolve by consideraion of the $\varphi - r$ combinations within the slant range and incidence angle limits.* The criteria and procedures for pattern selection from these ICA-altitude combinations are discussed in subsequent sections of the paper.

* Other limits peculiar to the system may arise. The emitted IR radiation of ascending rocket vehicles may be the information sought in a military surveillance system.[3] Here a "ground clutter" or background noise problem exists, so that ICA's take an annular form. The equipment is blind within the inner circle. This circle must not be so large that a vehicle may escape detection, and consequently the outer circle is delimited.

One may imagine other than circular ICA's produced by various equipment designs and operational techniques. It is felt they are of secondary importance because of the many factors leading to the circular ICA. However, if noncircular cases do arise, the following discussion may be taken as a model for their inclusion.

Pattern Criteria and Synthesis

We may now address the problem of satisfying a system coverage requirement with an assemblage of circular ICA's. Each ICA shows a time dependence governed by the orbital motion of its associated satellite. For an example, assume that the requirement is continuous in time and global in extent. That is, every location on the Earth must lie within an ICA at every instant after completion of the pattern. As a starting point, we assume the orbital motion to be purely Keplerian and regard all non-inverse-square forces as possible disturbing causes. If the required coverage area is symmetric with respect to the Earth's center, then ICA's of constant diameter for all satellites and within the orbit of each satellite should be employed. This is implied by the fact that the locus of sub-satellite points does not, in general, retrace itself. (The exceptional cases given in a recent paper[4] have a low probability of occurence.) Thus, orbits have no preferred locations within the latitudinal limits set by their inclinations over a time interval which is long compared to the orbital period. Since the area between these limits is also centrally-symmetric, a "principle of uniformity" may be invoked. This leads to the consideration of patterns made up of circular orbits of equal altitude. It should be emphasized that this applies only to the case of a centrally-symmetric coverage requirement.

A global coverage requirement may be satisfied by many patterns. Selection from these patterns involves consideration of diverse criteria. An attempt will be made to state the basic criteria and determine which should be used in the initial pattern selection and which should be assigned to more advanced phases of the system design process.

(a) Attainment Effort (costs)—a function of the number of satellites in the pattern, the equipment type and weight in each satellite, and the orbital altitude.
(b) Maintenance Effort (costs)—dependent on many of the factors in the attainment effort but with a strong statistical basis.
(c) Stability—essentially measured by the time interval from an initially satisfactory pattern to violation of the coverage requirements due to pattern perturbations—not associated with equipment reliability.
(d) Compatibility with present and future launching facilities and tracking ranges.
(e) Flexibility—the relative effort involved in adapting to other system requirements.

One quickly concludes from these criteria that (a) is the only one suitable for pattern initial selection. Items (b) and (c), though essentially quantitative in nature, require an assumed set of orbits to obtain numerical values; items (d) and (e) are essentially qualitative. It will be shown that use of (a) allows for a direct synthesis of optimal patterns so that design charts may be generated. These then serve as a basis for the consideration of the other criteria.

Pattern selection from a fixed ICA diameter implies prior specification of equipment coverage characteristics and operating altitude. The remaining attainment effort parameter, viz. number of satellites in the pattern, may then be minimized. The costs related to these minimum numbers can be compared with the costs involved in developing equipment with various ICA capability.

A "ring" is defined to be a group of equally-spaced satellites whose orbits differ only in the time of nodal passage. Fig. 3 shows the relative positions of the ICA's for such a group composed of m satellites. The ICA centers are separated by a central angle of $2\,\pi/m$ radians. If m is large enough so that this spacing is less than the ICA diameter $2\,\varphi$, a continuous strip of width $2w$ will be covered. The particular ring shown makes an angle α with the equator where α is also the orbital inclination of all satellites in the ring. Ω is the nodal longitude measured from some reference meridian.

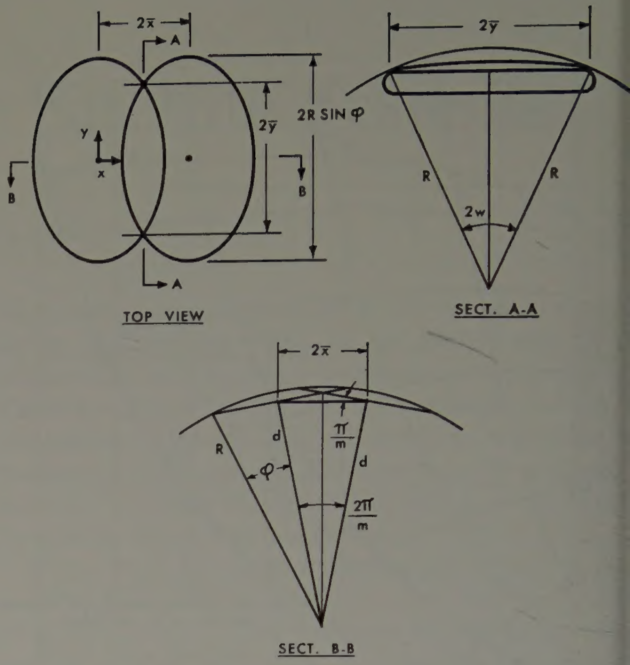

Fig. 4. Geometry of circular ICA overlap

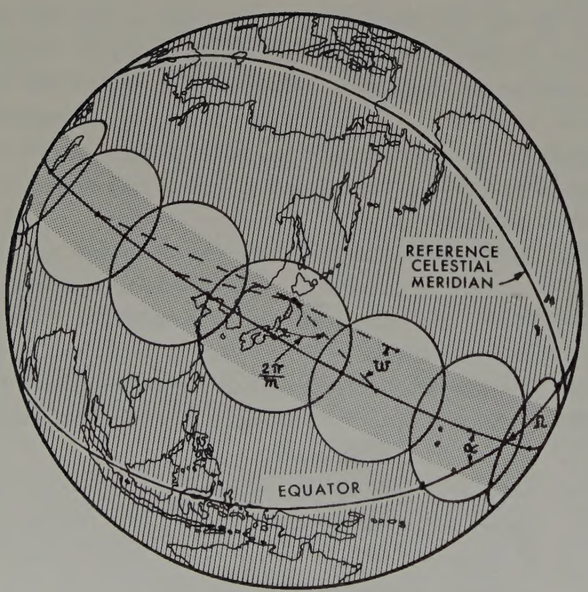

Fig. 3. Ring formation from equally-spaced ICA's

The ring half-width is a function only of φ and m. B
cause of the curvature of the Earth's surface, two ci
cular ICA boundaries, projected on a plane normal t
the local vertical at their line-of-intersection (A-A i
Fig. 4), will be ellipses with equations:

$$\frac{x^2}{R^2\sin^2\varphi\cos^2\pi/m}+\frac{y^2}{R^2\sin^2\varphi}=1, \qquad (1$$

$$\frac{(x-2\bar{x})^2}{R^2\sin^2\varphi\cos^2\pi/m}+\frac{y^2}{R^2\sin^2\varphi}=1. \qquad (2$$

R is the radius of the Earth. From the figure, d =
$R\cos\varphi$; therefore, $\bar{x} = d\sin\pi/m = R\cos\varphi\sin\pi/m$. Th
ellipses intersect at the points, $x = \bar{x} = R\cos\varphi$ si
π/m, $y = \pm\bar{y} = \pm R\,(\sin^2\varphi - \cos^2\varphi\tan^2\pi/m)^{\frac{1}{2}}$, so tha

$$w = \sin^{-1}(\sin^2\varphi - \cos^2\varphi\tan^2\pi/m)^{\frac{1}{2}}$$
$$\equiv \sin^{-1}\beta\,(\varphi,m). \qquad ($$

Now rings may be assembled to effect the require
global coverage in two ways. First, several rings of th
same inclination but different nodal longitudes may b
combined to extend coverage longitudinally. Secon
the nodal longitudes may be held fixed, but their in
clinations varied to provide greater latitudinal coverag
A pattern which obtains global coverage by the firs
method we class as an L-type. If the second is employe
a λ-type pattern results. It will be shown that rath
small differences exist between the minimal number
satellites for each type; certain operational considera
tions dominate any selection.

To determine the optimal patterns for each type, he critical region of least overcoverage must be ivestigated. In the case of the L-type, a ring inclination of $\alpha = \pi/2 - w$ is prescribed. The number n, of uch rings required depends on the sector of the quator corresponding to each ring. From Fig. 5,

$$\sin \sigma \frac{\sin w}{\sin \alpha} = \tan w. \tag{4}$$

etting N_L be the minimum total number of satellites equired and noting that $2\, n\alpha \geqq \pi$ for complete coverage round the equator,

$$N_L = m\, n \geqq \frac{m\, \pi}{2 \sin^{-1} \tan \sin^{-1} \beta(\varphi, m)} . \tag{5}$$

learly this lower bound is defined only at values of and m for which $\tan \sin^{-1} \beta \leqq 1$, i.e. $w \leqq 45°$. When $= 45°$, eq. (4) implies $\sigma = 90°$, and the two types of atterns coalesce.

The critical region for λ-type patterns is along the elestial meridian of anti-nodes common to all rings. gain, if n is the number of rings, $2nw \geqq \pi$ gives lobal coverage, and

$$N_\lambda = m\, n \geqq \frac{m\, \pi}{2 \sin^{-1} \beta(\varphi, m)} . \tag{6}$$

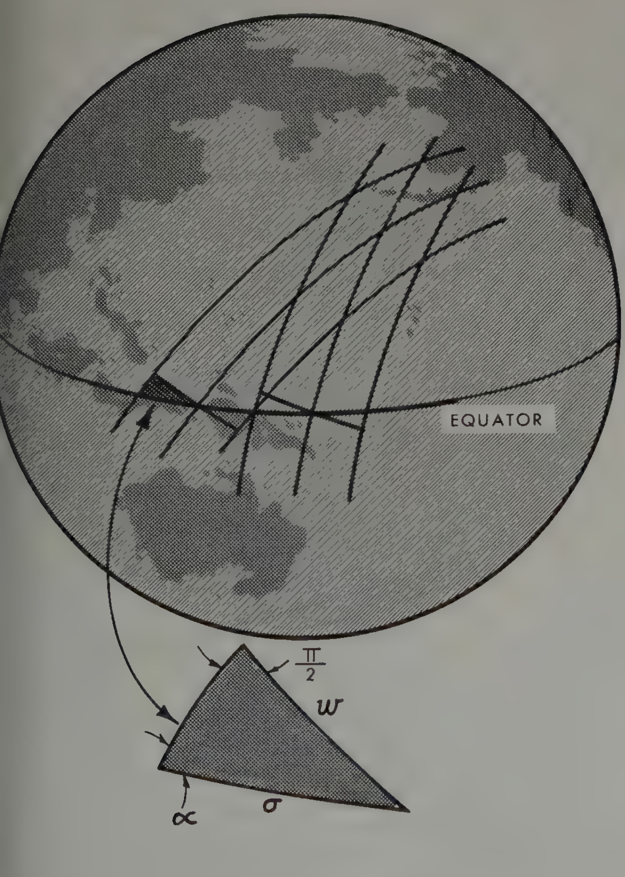

FIG. 5. Local geometry of equator for L-type patterns

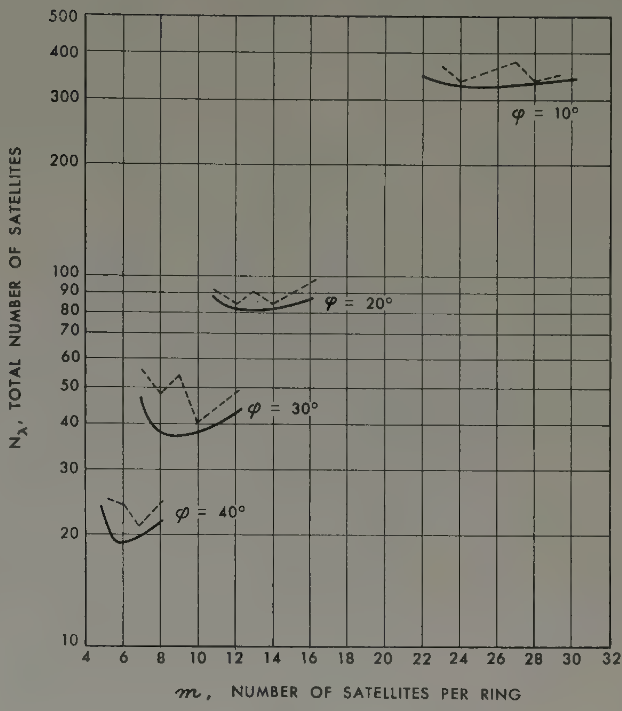

FIG. 6. L-type pattern optimization

Comparing eqs. (5) and (6), we see that the lower bound for N_L is less than the bound for N_λ. The full mathematical formulation of the problem of determining optimal patterns must include the integer property of m, n, N_L and N_λ. Relations between m and n must also be satisfied, viz.

$$n \geqq \frac{\pi}{2 \sin^{-1} \tan \sin^{-1} \beta}, \tag{7}$$

and

$$n \geqq \frac{\pi}{2 \sin^{-1} \beta}, \tag{8}$$

for L- and λ-types respectively. A straight-forward solution process consists of exploring values of N_L and N_λ for integer combinations of m and n which satisfy eqs. (5–8) regarding φ as a parameter. When this is done the dashed curves of Figs. 6 and 7 result. It is interesting to compare these with curves obtained by assuming the variables to be continuous. These are shown as solid lines in the figures. The lowest point of the dashed curves gives the optimal pattern variables. In some instances these variables are not unique. When $\varphi = 40°$ for example, an N_L of 18 results from $mn = 6 \times 3$ and 9×2. Both arrangements meet the overall coverage requirement with the same minimum total number of satellites. Fig. 8 gives N_L and N_λ as they depend on φ in the range $10° \leqq \varphi \leqq 40°$. Because of the relatively large magnitudes of N_L and N_λ in this range, a smooth curve through several computed points allows intermediate values to be estimated quite accurately. When φ is greater than about 40°, however, the step-wise character of the minimal numbers needs exact representation.

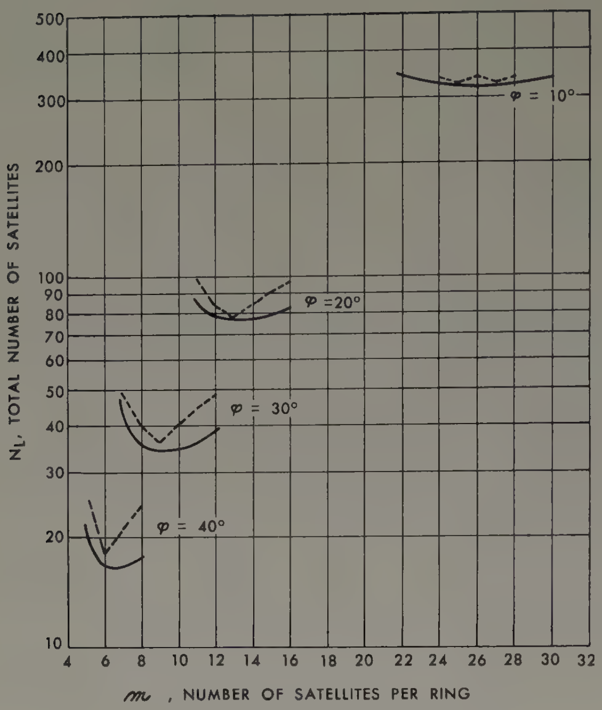

FIG. 7. λ-type pattern optimization

TABLE 1

Optimal Pattern Variables for Continuous Global Coverage

φ Circular ICA Half-Central Angle, degs.	Pattern Type	w, Ring Half-width, degs.	m., Number of Satellites per Ring	n, Number of Rings,	N, Tota Number o Satellite
10	L	7	15	13	325
	λ	7	24	14	336
		8	28	12	
20	L	15	13	6	78
	λ	14	12	7	84
		16	14	6	
30	L	22	9	4	36
	λ	24	10	4	40
40	L	28	6	3	18
		35	9	2	
	λ	32	7	3	21
50	L	37	5	2	10
	λ	37	5	3	15
60	L	44	4	2	8
	λ	52	5	2	10
70	L, λ	46	3	2	6
80	L, λ	69	3	2	6
90	L, λ	90	2	1	2

The process of integerizing the solutions produces some ring overlap at the critical regions even for optimal values of the variables. This degree of overlap depends on the parameter φ as would be expected. Subsequent sections of the paper will consider the manner in which this overcoverage effects pattern stability, attainment accuracy, etc.

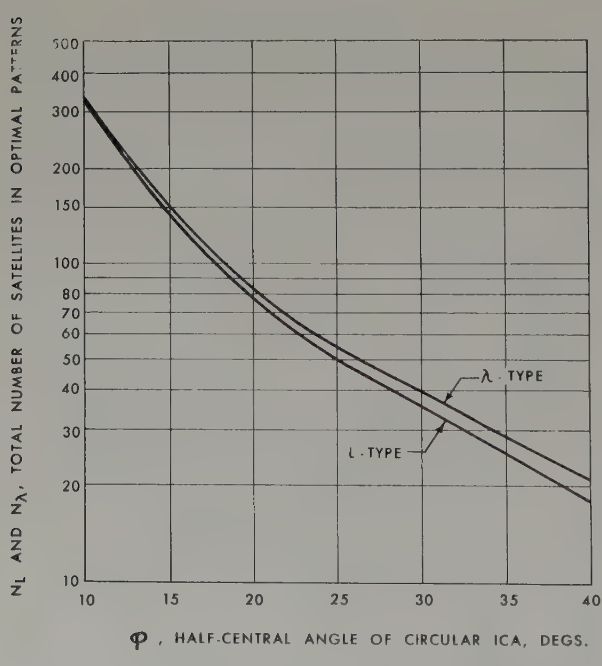

FIG. 8. Minimal number of satellites vs. ICA size

Table 1 gives the values of the pattern variables f minimum total number of satellites for $\varphi = 10°$, 2(30°,· · ·, 90°. Between $\varphi = 60°$ and 70°, w for optim L-type becomes greater than 45°. Although the tw pattern classes lose their identity at this point, t manner in which the intrinsic overcoverage is distri uted retains their features. When, for example, φ 80°, we may place both rings at an inclination α = 9 $- w = 21°$ to just cover the poles and obtain doub coverage in the equatorial region. On the other han the inclination may be raised to $\alpha = w = 69°$ to pr duce overcoverage in the polar regions. The lines-c nodes in each case are coincident.

The principles used for global coverage are readi extended to cases of restricted coverage, say, betwe an upper latitude λ_u (small circle B), and a lower la tude λ_l (small circle A). As $\lambda_u - \lambda_l$ decreases, the r quired number of satellites for both types of patter is reduced in general. The photographs (Figs. 9 and 1 are intended as an aid in the vizualization of types pattern coverage. On both, solid white lines indica the trace or locus of sub-satellite points on a fix Earth. The boundaries of different rings carry differe markings. The two patterns shown are directly co parable, i.e., the areas of required coverage are t same ($\lambda_u = 75°$, $\lambda_l = 30°$). The degree of overlap, Δ may be considered due to the integer constraints on t variables of required overcoverage demanded by oth factors.

Fig. 9. Typical L-type pattern

Fig. 10. Typical λ-type pattern

Pattern Stability

A full treatment of pattern stability is beyond the scope of the present paper. One may quite easily delineate the cause-effect relationships however. Some quantitative conclusions can be made which supplement the heuristic value of such an approach.

As previously defined, pattern stability may be related to the time interval between completion of the pattern and loss of required coverage due to non-Keplerian forces (not accounted for in the model) which give rise to differential motion of the ICA's. The definition is extended to include the effect of placement inaccuracies. Thus departures from the specified or nominal orbit elements during pattern attainment may in time cause "holes" to open in the coverage area. Table 2 has proved useful in this regard.

TABLE 2
ICA Variations

Cause	Effects on ICA: Positional, Along Orbit Track	Effects on ICA: Positional, Across Orbit Track	Effects on ICA: Size
Δa	+ or −, ∞	———	+ or −, 0
Δe	+ or −, P	———	+ and −, P
$\Delta \alpha$	———	+ and −, P	———
$\Delta \Omega$	———	+ and −, P	———
ΔT_Ω	+ or −, 0	———	———

(——— denotes no effect)

Since the argument of perigee is not defined for a circular orbit, we may examine the effect of off-nominal values in the other five elements on the ICA position and size. These elements are taken to be:

a, semi-major axis
e, eccentricity
α, inclination
Ω, longitude of the ascending node
T_Ω, time of nodal passage from the epoch

If some element, say α, has a variation $\Delta\alpha$, then a "+ and −" effect indicates bi-directional changes in the ICA position or size irrespective of the sign of $\Delta\alpha$. A "+ or −" effect depends on the direction of the cause. The symbols 0, P, ∞ denote the period of the effect. They correspond, in order to a bias shift, an orbital periodic variation and a steady drift. One notes that only Δa produces a drift effect and that only in the ICA position along the track. All others may be tolerable due to the intrinsic overcoverage of an optimal pattern or "designed out" by the additional overcoverage of an off-optimal pattern.

In addition to placement inaccuracies, gasdynamic drag, higher harmonics in the Earth's gravitation field and third-body perturbations will alter the orbits. It is emphasized at this point that orbital changes *per se* do not degrade the coverage capability of a pattern. What is important here is a "differential pertubation that acts differently on satellites to change the relative positions in the pattern.

The primary perturbations of a two-body orb arising from the Earth's oblateness are as given b King-Hele and Merson:[5]

(1) secular rotation of the line of nodes
(2) radial excursions of half orbital period
(3) dependence of nodal period on the inclination
(4) secular rotation of the line apsides

The last has little significance for nominally circular o bits; the third has no effect since ring geometry is invar ant with a change in period common to all satellites i the ring; the second is of quite small magnitude and ca easily be designed out. The regression of the line c nodes, $\dot{\Omega} \doteq 10\ (R/a)^{3.5} \cos\alpha$ degs./day, has no effe on L-type patterns since all satellites have the same and α. The pattern merely "sees" the Earth rotat more rapidly than once per day if direct orbits a employed. The rings of λ-type patterns however, wi experience a differential nodal shift and in general, hole will open in some finite length of time. Som preliminary computations indicate lifetimes of th order of months for optimal patterns although th results are insufficient to serve as a basis for an general conclusions. We note this disadvantage c λ-type as compared with L-type patterns and, at th same time, observe that off-optimal λ-type patterr could increase their lifetime indefinitely.

A study of the influence of celestial bodies such a the Sun and Moon (and perhaps other artificial satellite as well) on pattern stability lies in an area of investiga tion noted for its difficulty and scarceness of quantita tive results. Without attempting to describe the preser status of the problem of a multiple-body system unde Newtonian forces, one notes that even in the restricte three-body case, no complete solution is availabl If linearity is assumed so that the separate effects o pattern orbits from the presence of other bodies ca be added, the conditions of the restricted three-bod problem are met for each perturbing body in tur If, further, the most adverse constellation of the Eartl satellite-perturbing body system is supposed, "bounds may be placed on pattern lifetimes. This approac essentially ignores the dynamics of the problem an is useful only in estimating the maximum differenti acceleration two adjacent pattern satellites migl experience.

Gradecak's work[6] would indicate the existence c secular perturbations only the node and the argument c perige—the latter defined on the perturbed orbit wit periodic eccentricity variations about the zero poin The magnitudes of these drifts are exceedingly sma for near-Earth ($a/R \leqq 2$) satellites amounting to a fe minutes of arc per year. An additional theoretic resu is based on the recognition of "resonant orbits" whic tend to amplify the effect of these small perturbation

satellite orbit of semi-major axis $a = a_s$ is resonant with, for example, the lunar orbit $a = a_m$ when the equation $\nu a_s^{3/2} = \nu' a_m^{3/2}$ has integer solutions (ν, ν'). Values of a_s which admit integer solutions are to be avoided, especially if these solution pairs are composed of small integers. These orbits for small ν and ν', say each an integer less than ten, are beyond the distances one might expect to attain the limiting patterns of practical significance, namely those of three satellites. Lower reasonant orbits correspond to very high order terms in the perturbation function. These amplified variations would be exceedingly slow.

The gasdynamic decay of equal-altitude circular orbits does not disturb the ICA relative positioning. Apart from a consideration of pattern lifetime based on the time to re-entry, the tendency to reduce the ICA size and hence the degree of coverage needs investigation. Using Peterson's data[8] and values of $d\varphi/dr$ for $i \doteq 90°$ (the worst case) from Fig. 2, the ICA radius decreases on the order of 10^{-1} and 10^{-6} degs./yr. for altitudes of 250 and 1000 miles respectively. The effects of these rates are clearly ignorable.

Operational Considerations

Congressional testimony[9] has brought out some of the operational complexities of coverage-type satellite systems. Of primary importance is the feasibility of placing a satellite in orbit with prescribed nominal values on all six orbital elements. This, of course, relates to the rendezvous (with a hypothetical satellite) problem which has received considerable attention recently. Let us suppose the availability of a single launch site and booster vehicle which give a capability of meeting the required semi-major axis, eccentricity and inclination values for a given ring. These elements are independent of the time of launch; the other two are not. Once each day the site is in position to match the nominal nodal longitude of the ring by using the same reference ascent path. The in-ring spacing between two satellites launched by the above procedure would vary according to the period of the orbit. Now if this period divides into the day with a $1/m$ remainder, m satellites can be launched on successive days so that their in-ring spacing is uniform. If an attempt fails, one is assured of another opportunity m days later.

The above method ignores nodal regression. Some dispersion in nodal longitude would result if it were strictly followed. At satellite altitudes greater then one Earth radius, the nodal dispersion would be less than one degree for each day after the start of the launch sequence. Although this value does not appear to be very large, a certain amount of overall readiness is necessary to establish rings by this method. Nodal dispersions of the magnitude implied by, say, a week of daily launchings can be reduced with quite small out-of-plane thrust corrections at the antinodes of the orbits. These post-placement corrections may be unnecessary if the choice of nominal elements produces sufficient overcoverage. The number of satellites over optimal to produce the type of overcoverage associated with nodal dispersions is small compared to that required when the in-ring spacing is ignored. Also schemes for correcting the latter errors[10,11] have comparatively large velocity and time requirements. It appears, therefore, that patterns with the aforementioned divisibility property have an operational advantage. We will investigate the possibility of meeting this condition by considering a numerical example.

Fig. 2 gives for $i = 90°$, $\varphi = 60°$ at $r = 2.0$ Earth radii. From Table 1, m is 4 and 5 for optimal L- and λ-type patterns respectively. The satellite period is 0.164 days resulting in 0.10 revolution shift per day. This shift is less than the $1/m = 0.25$ and 0.20 necessary. An altitude adjustment of less than 100 n.m. will produce the proper phasing. This is considered to be within an actual design envelope relating altitude to ICA. Other examples would, of course, require greater percentage adjustments, but the model allows for much give-and-take so that a developable preliminary design of the system is assured.

The launch sequencing and ascent guidance problems L-type patterns are not as difficult as those for λ-type since all rings are at the same inclination. The same ascent path relative to the ground tracking and guidance instrumentation may be used. This factor may be outweighed by differences in what might be termed the intrinsic overcoverage area. As Figs. 9 and 10 show, even optimal patterns have regions on the Earth in which multiple coverage is maintained continuously. This region is in higher latitudes for L-type and lower latitudes for λ-type. A choice may well depend on which region could least easily suffer a loss of coverage in the event of equipment failure.

The great costs associated with satellite systems of the type described leads one to assume that their "reason for being" will be of first-rank importance to a nation's political and military position. It is difficult to imagine such expenditures tied to other than determinate and well-defined system requirements and execution. One must not take the specification in probabilistic terms of, say, an ICBM system accuracy (CEP) as a counter-example to the foregoing argument. Here perfect execution results in perfect attainment of the objective. The existence of CEP-type specifications merely implies the realization that perfection is ethereal. On the other hand, a probabilistic or so-called random pattern degrades the purpose even though the development and execution of the system were thoroughly and exactly carried out.

Acknowledgement

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The Editor

Astronavigation—Guidance and Control in Space[1]

Robert E. Roberson[2]

Abstract

Within the last few years there has been an increasing recognition of the central role of guidance and control in astronautics and a marked increase in the number of studies of such problems. The bulk of this literature sometimes makes it difficult to see how individual efforts fit together, but it is generally agreed that there is a legitimate discipline—under whatever name—which can be characterized loosely as "space guidance and control". I propose to call this "astronavigation". The purpose of this paper is to dissect out the ingredients of astronavigation, to show what problem areas are logically encompassed by the whole disciplinary area, and how these fit together. The paper is in no way a survey of the literature of the field—the examples chosen to illustrate certain problems are often taken from the relatively early literature on those topics. However, the topical breakdown itself provides a convenient framework for such a survey.

Introduction

The term "astrodynamics" has been applied by Herrick [1] to trajectory and orbit analysis of astronautical systems. Boni [2] coined the term "sidar-mechanics" to apply to the same area, but his terminology has not been accepted. This distinction from classical celestial mechanics is useful, but it is convenient to recognize six distinct facets: three direct or analysis problems and three inverse or synthesis problems.

Celestial Mechanics

The first direct problem is to analyze the motion of a system under a completely specified set of non-propulsive forces following a specified initial state. This can be considered to be purely celestial mechanics, somewhat enlarged from the traditional discipline to include dissipative and other nongravitational forces. It includes selection of basic reference systems (see [3] and remarks in [1]), examination of the values of astronomical and gravitational constants [4], and the methodology of analytical and numerical investigations of trajectories and orbits.

[1] Portions of this work were supported by Republic Aviation Corporation.

[2] Systems Corporation of America—Los Angeles, California.

[3] In truth, the vehicle may be a collection of a number of rigid bodies and may have many degrees of freedom. However, we speak here as if it were a single body.

Analytical attacks have been made on the effects of a number of perturbations which were not of much classical interest. Two that have received most (and probably disproportionate) attention are the effects of planetary oblateness and air drag on the orbit of a close satellite. The methods have been largely classical, such as variations of parameters and other perturbation methods, and the goals normally have been to derive the low order secular behavior of the orbit elements.

Numerical work has been of two general types, clearly distinguished by Walters [5] as concerning "design (or feasibility) trajectories and precision" trajectories. The importance of precision trajectories also has been emphasized by Herrick and Baker. The present author [6] has commented on the domains of special value of numerical and analytical approaches and has observed that a fruitful area of further progress should be that in which these are combined in a single study to exploit the special advantages of each to the fullest.

This area of celestial mechanics also should be understood to include variations about free-flight trajectories, particularly those resulting from changes in initial conditions. The literature is replete with developments (and redevelopments) of the effects of initial position and velocity variations upon the elements of two-body orbits or upon certain terminal conditions at a subsequent point on a two-body orbit, topics which have direct and important implications to guidance and control. There has been less attention to the terminal effects of initial errors following other kinds of trajectories (e.g. three-body trajectories), for these must be obtained by numerical computer studies, but this situation has changed rapidly during the past year. Most results of this kind are available only as company reports, but one published example is the earlier work of Lieske [7].

The celestial mechanics aspects of astrodynamics are particularly important in analyzing certain operational aspects of space vehicle applications some of which bear indirectly upon the problems of guidance and control. However, it is in the question of variations about or deviations from the nominal that the celestial mechanics type studies impinge most directly on guidance and control.

The inverse problem that corresponds to the analytical area of celestial mechanics is that of using observations on orbits or trajectories to determine certain parameters in nature, such as Earth's oblateness and atmospheric properties. This kind of problem has been discussed in general terms in the literature but has become much more pressing with the accumulation of data on existing satellites. It seems to be a somewhat more difficult problem than originally expected, especially when observations are of indifferent quality. Extracting "average" properties such as a smoothed atmospheric density vs. altitude relationship is possible, but details of atmospheric structure (including diurnal and latitude variations, atmosphere rotation, and the like) and of other dissipative effects are well masked by gross air drag and nonconservative perturbations, and a forteriori by observational errors. An example of what can be done in this problem area when accurate element determinations are available is given by recent work of King-Hele [8]. There is no need to dwell on this topic, but it illustrates a definite region within which astrodynamics and guidance may be relatively unrelated problems and corrects the occasional implication that astrodynamics *is* space guidance.

Maneuvers

The second part of the trajectory and orbit problem is concerned with powered maneuvers. Given the basic operational concept for a certain astronautical mission, what kind of nominal path *should* be followed, and what kind of thrust or steering program is needed so that it *will* be followed under nominal conditions? Here again we have the distinction between analysis and synthesis, but in this case the analysis part is relatively straightforward and the problem of synthesis tends to be primarily important.

The synthesis problem also may be called the problem of the "maneuver concept". I have elaborated the role of this topic in establishing guidance and control requirements in [9]. Fundamentally, the formulation of a maneuver concept is deciding what kind of maneuver sequence is "best" used to accomplish the mission goals (where the system is to go and why) within the limitations imposed by the system's physical nature, including such factors as rocket engine characteristics and available propellant weight. For the case where a high quasi-impulsive rocket thrust is used, this means a device of impulse directions and magnitudes for a set of thrust periods separated by periods of free-flight.

This situation is more difficult in cases where the thrust is not quasi-impulsive, then there are two major subcases, one relating to high thrust boost and the other to very low thrust space flight where propulsion is by ion rocket or similar means. The former is a problem which has been treated rather extensively, often by methods of the calculus of variations (for example, in numerous works of Miele). Although the low-thrust case has been approached this way (e.g. in [10]), the choice of a nominal low-thrust steering program probably is better done on the basis of resulting guidance and control simplicity than on traditional propulsion or time considerations. The author inclines strongly toward this point of view and has heard it well expressed by Levin [11] and others.

This comment introduces the central difficulty of developing a maneuver concept, namely the choice of a criterion of merit. What is meant by "best" maneuver or steering program? Classically, this has been taken to be a propulsion (characteristic velocity) minimization. Typified by many of the excellent papers of Lawden (e.g. [12]–[14]), this kind of criterion is particularly appropriate where propulsion capabilities are marginal. However, there are many other factors too, such as guidance and control practicalities and operational considerations, which can become dominant in certain cases. For example, even the "simple" case of two-impulse maneuvers becomes complex when multiple "payoff" factors, not necessarily consistent, arise. In [15] the author discusses the selection of an ascent trajectory for satellite rendezvous where there is a tradeoff between characteristic velocity and expected waiting time, an operational factor sometimes of extreme importance.

Many other examples can be quoted to illustrate the deficiencies of the usual single-variable criteria. However, it suffices to say that there is increasing recognition of the operationsl advantages of certain classically "non-optimum" trajectories, and further developments can be expected along these lines. At the moment, the case of flight throught the atmosphere probably is best under control while that of low-thrust space flight is least.

Low Thrust Maneuvers

Because of the importance of the low thrust maneuver problem, it is examined further here with reference to existing literature. Consider first the direct problem of determining the path when a steering problem is prescribed. The prescription, in this case, takes the form of a rule which relates the orientation of the thrust vector to time and to such observables as the instantaneous vehicle position and velocity.

Several basic problems of the direct type have been mentioned time after time in the literature. Ordinarily it is assumed that thrust is directed either radially (i.e. along the line from the vehicle to the gravitating center), circumferentially (i.e. perpendicular to above and in plane of travel), or tangentially (i.e. along the forward velocity vector). Assumed steering programs invariably are one of these or some combination of these with a change from one type to another at chosen times and with thrust levels which are piecewise constant or which vary in some special fashion which permits closed form solutions of the dynamical equations. Past studies have had one of two basic goals. Either they have been directed toward finding the path or related characteris-

TABLE 1

Literature on the "Direct" Problem of Low Thrust Trajectories

Thrust Direction	Purpose	
	Path, Element Changes and Related Topics	Propulsion and Related Topics
Radial	11, 16, 17, 18, 19, 20, 21	16, 17, 22
Circumferential	11, 16, 17, 23	16, 17, 22, 23, 24
Tangential	23, 25, 26, 27, 28, 29	22, 23, 24, 27, 30

tics (such as breakaway time, changes in orbit elements) or they have concerned some aspect of propulsion (such as characteristic velocity). Table 1 summarizes some of this literature on the direct problem in terms of these characteristics. Some of the references have an approach which is purely analytical, others an approach which is purely numerical, but most involve a mixture of analysis and numerical computation.

Next consider the inverse problem of determining a nominal steering program when the origin and destination are assigned and when all of the dynamical aspects of the motion can be described completely, because, there are no unpredictable or error terms in the equations of motion. A related problem is to determine the steering program which results in specified changes in orbit elements.

There have been three basic approaches to this problem. One has been exploratory. A steering program is chosen which is "intuitively natural" and a resulting trajectory is computed. If the terminal conditions are not those desired, the parameters of the program are changed slightly. For example, the duration of steering in one direction could be prolonged at the expense of a subsequent steering mode. Or, the thrust level might be readjusted. Then the trajectory is recomputed and the terminal conditions are again determined. It is hoped in this process that judgment of the investigator will be an adequate guide so that the procedure converges upon an acceptable trajectory. In the problem of changes in orbit elements, the required changes are usually assumed small so that analytical approximation methods apply. Most of the tentative or preliminary design type low-thrust trajectories obtained heretofore seem to have resulted from this method. Table 2 tabulates the pertinent literature.

TABLE 2

Literature on the "Inverse" Problem of Low Thrust Trajectories

Maneuver or Destination	Thrust Direction	Sought	References
Radius change	tangential	transfer time vs. acceleration	26
	circumferential and radial	steering mode, switching point	15
Eccentricity change	intermittent, tangential	time to change vs. acceleration level and thrust duration	26
	radial	acceleration level	18, 19
Various	various, including some thrust, magnitude and variations	exploration of effects	31
Mars	tangential circumferential	acceleration vs. time	23
Moon	tangential	reversal points	28

A second method for determining steering program is based upon the ingenious investigations of Rodriguez [32]. He has observed that at least for the case of transfer to a circular terminal orbit, the initial and terminal kinematic states can be represented as single points in an energy-momentum phase diagram. It follows that any path of states in this phase plane followed by the vehicle which contains the initial and terminal point is a steering program guaranteed (in the absence of deviation-producing errors) to result in a proper trajectory. Rodriguez shows how steering programs can be built up naturally from constant energy and constant angular momentum segments of trajectory, together with other segments of a "natural" type. (E.g. with thrust in the direction of the velocity vector or thrust normal to the radial vector between the vehicle and gravitating center.) In this procedure it is not possible to reduce the problem entirely to graphical and analytical means, since certain quantities of interest still can be obtained only by machine solution of differential equations. However, the Rodriguez method is a convenient framework for visualizing the effect of changes in the steering program and of systematizing the exploratory type investigations. It is felt to be worthy of considerable further study, with particular emphasis on the representation of the effects of errors within the same framework.

A third procedure might best be called the "rational" method, consisting of the generation of a steering program by a purely formal means after setting up the problem as one in the calculus of variations. All approaches of this class have certain things in common. First, one must have an explicit criterion for selecting the trajectory. Second, one must establish mathematically all of the essential constraints on the problem. Third, the variational equations, which tend to be much too complicated for analytical treatment, must be formulated in a way amenable to machine computation. Fourth, one must investigate the sensitivity of the resulting solution to changes in parameters and see whether the maximum or minimum under the criterion chosen is a sharp one or a broad one. Fifth, one must reconcile the resulting steering program with control mechanization practicalities. Published investigations of this class seem to be those of Lawdon [13], [14] and Faulders [10]. Lawdon uses a minimum energy criterion, Faulders a minimum time criterion. The major difference otherwise is that Lawdon focuses his attention primarily upon the calculus of variations formulation

while Faulders gives considerable attention to the machine solution problem.

It is seen from the above commentary on previous work that most previous results concern the direct problem of finding trajectories numerically, while some investigations have been made of the inverse problem determining a steering program given the end conditions. However, our state of knowledge about the inverse problem is still rather sketchy and considerable additional work is required.

Attitude Dynamics

Heretofore, "astrodynamics" has been used exclusively as referring to the path motion of a space vehicle—the motion of its center of mass. However, space vehicles actually have six degrees of freedom, the additional three degrees of freedom representing rigid-body motion about the center of mass.[3] Dynamical problems associated with rotational or attitude motion are completely analogous to those discussed above for translational or path motion. There is no rational basis for failing to include them under the heading "astrodynamics", in spite of the fact that it has not been conventional to do so. Clearly, trajectory and attitude dynamics are intrinsically interrelated during powered flight phases, and questions of attitude dynamics are implicitly encompassed by the previous area of "maneuvers". Table 3 tabulates astrodynamics problems.

TABLE 3

Astrodynamics

Major Areas	"Direct" Problems	"Inverse" Problems
Celestial Mechanics	a) perturbation analysis by analytical approximation (b) numerical compilations of "feasibility" and "precision" trajectories (c) effects of variations about free-flight path	Inferring trajectory characteristics or physical parameters from observation of path
Maneuvers	(a) determination of trajectory given thrust and steering program (b) effects of variations about powered-flight path	Development of a maneuver concept: (a) impulse sequence (b) powered flight in atmosphere (c) low thrust space flight
Attitude Dynamics	(a) dynamical equations of motion (b) derivation of torques (c) free-body motion analysis	Inferring parameters from observation of attitude

On the other hand, the attitude-dynamical analogues of celestial mechanics are essentially new to the discipline of astrodynamics. They could be subsumed under "celestial mechanics" by appropriately generalizing that term. However, I believe that a new name for this area is justified and propose simply "attitude dynamics", understanding by this term free-flight attitude dynamics.

Problems of attitude dynamics are effectively the "foundation problems" of attitude control which have been outlined and discussed in [33] and [34]. Such topics are included as the choice of suitable attitude reference frames, the development of the basic dynamical equations of attitude motion (non-trivial if the space vehicle contains moving parts), the derivation of torques acting on the vehicle from external sources (or internal, other than torques deliberately applied for control), and the behavior of spin-stabilized satellites.

These are problems of the "direct" type, but in principle the "inverse" problem exists as well. For example, one might imagine inferring something about atmospheric or electromagnetic field parameters or other dissipative effects by observing the damping of rotational motions. Such damping has been analyzed to some extent (see references in [34]), particularly as it applies to spinning bodies in the Earth's magnetic field, but principally from the "direct" point of view. Probably one reason why the "inverse" problem is poorly represented is the observational difficulty attached to it, unless the vehicle is specially instrumented with these effects in mind, compared with observations of position.

In summary, the problems of free-flight attitude dynamics represent an important, new, but rapidly growing area which properly should be included under the heading "astrodynamics".

Astronavigation

Terminology

Even if everything behaves perfectly as predicted, there are still guidance and control problems, for the vehicle must be made to follow its nominal path with its nominal attitude. In fact, though, things do not behave perfectly and deviations from nominal conditions are bound to occur. It is not enough to precompute steering or torque signals needed to maintain nominal conditions—overcoming the effects of deviations is the very essence of guidance and control.

Thus far I have spoken of "guidance and control" to describe a certain problem area, and the reader probably has understood in a general way what was intended. However, I submit that neither term is quite adequate to characterize this field, while both together are awkward. What one is really trying to accomplish by the operation of "guidance and control" is to cause the vehicle to pass through a set of kinematic states which satisfy a prescribed set of terminal and perhaps en route conditions. A "kinematic state" in free flight is

described by any set of parameters which permits the prediction of subsequent motion, such as initial values of position, velocity, orientation and angular velocity. Or, in some cases position and velocity may be replaced by a set of orbit elements. We may think of the kinematic state as constant during any period of free flight, changing during any powered maneuver. (See [13]).

To see that this is more than just "guidance" or "control" in the traditional sense, consider Figure 1 which shows schematically the processes involved in carrying the system through a set of kinematic states.

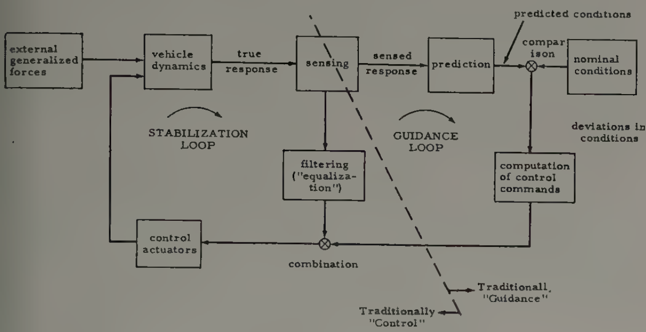

FIG. 1. Generalized space vehicle control operations

This is a generalized diagram in the sense that it represents a multi-variable control system. The vehicle is acted upon by generalized forces (i.e. forces and torques) and its behavior in terms of some set of variables is sensed. This may be position and/or velocity relative to some convenient coordinate system, orientation or orientation rate, or any other physical quantities which characterize kinematic states. In one case, these quantities are operated upon by generalized filtering and the resulting signal is fed to control actuators to stabilize the entire closed loop control system. Generally the purpose of this during powered flight is to keep the vehicle attitude from becoming unstable. This portion of the overall control of kinematic state is sometimes called the "inner loop" or might be called simply the "stabilization loop". All of the portion of the closed loop system shown in Figure 1 to the left of the dashed line is traditionally the "flight control system". The portion consisting of the actuators, "equalization" block, and corresponding sensing operations is sometimes called the "flight controller". Note that for free flight attitude control, this loop may constitute the entire control system.

On the other hand, the sensing operation also supplies outputs which are used for calculating or predicting certain "conditions", such as terminal position and/or velocity, orbit elements, or other similar factors. These "conditions" are whatever is needed to characterize the thing that the operation of path control is trying to accomplish. Predicted conditions are compared with nominal conditions and the differences or deviations are used as the basis for computing control commands. In general the latter consist both of steering signals and off-on signals. The control command signals are combined with the output of the equalization block and the result used as an input to the control actuators. The portion of the control system consisting of the prediction operation, comparison, and generation of path control commands is traditionally called the "guidance system". It is labeled "guidance 'loop'" in Figure 1, although "loop" strictly is a misnomer—it is not a closed loop until it is made so through the control system.

It is clear from Figure 1 that while there is a rational basis for the traditional distinction between guidance and control, these really have an inherent unity. They are just different loops in an overall control system, and the exact point of dichotomy is to some degree arbitrary. I believe there are advantages in deemphasizing the traditional division and speaking of the entire closed-loop system as simply "control", immediately dividing it into "path control" and "attitude control". This neither abandons the former terminology, nor raises problems by extending its meaning in unfamiliar ways. The teminology of attitude control is already well accepted, and one merely adds "path control" as a preferred substitute to "guidance and (flight) control".

Note that it is in the external force, vehicle dynamics and nominal conditions blocks, and possibly in the prediction block, that astrodynamics enters the picture. The "control" area, as I have proposed it here, would use the astrodynamical results as essential background but would be itself concerned with the rationals under which the operations in the various blocks are performed, the physical structure of these blocks, and the closed loop characteristics of the entire combination.

There is the problem of what to call the entire area that is neither entirely astrodynamics nor entirely control. I propose "astronavigation" as a suggestive term which seems suitable despite a possible confusion with the interpretation "navigation by reference to the stars". It usually begs the question to refer to the dictionary, but one does find there that "navigate" is to direct one's course or steer, while "navigation" is the act of navigating using calculations of position and direction, etc. This is exactly what we do mean by the operation we are discussing here whose major ingredients are astrodynamics and control.

Path Control

The foundations for path control are laid with the establishment of a maneuver concept which results in a nominal flight path. However, this astrodynamical question is only a first step, and beyond it there lie five areas of investigation which are unique to the path

control problem. The first is a careful examination of the nature of the disturbances acting on the vehicle to cause it to depart from its programmed trajectory. The second is a study of possible methods that might be used for sensing its deviation from the desired path or its absolute position and velocity relative to a prescribed reference frame. The third is a consideration of the characteristics of the devices that might be used to change its path, the actuators. The fourth is the development of a correction philosophy, which is not to be confused with the maneuver philosophy chosen in the absence of any error considerations. The fifth is a performance analysis in terms of the various sources of error that may be expected.

The disturbance analysis is important for several reasons. The qualitative nature of the disturbances can influence the collective action against them or impose specifications on the operation of certain parts of the system. For example, in a low thrust system one may find that major disturbances arise from random stochastic variations of line of thrust relative to the vehicle frame. The best remedy in this case might be to use the disturbing mechanism also as a corrective mechanism or to tighten the control of the thrust vector to a point where the stochastic effects are negligible. The quantitative magnitude of the path deviations resulting from the disturbances is important both as it effects the choice of sensors and a control concept. As an example, if no disturbances act to carry the vehicle out of its nominal plane of motion it may be possible at certain levels of accuracy to dispense with sensors for detecting such out-of-plane motion, or at least to give them a rudimentary form. If path deviations resulting from disturbances are small, but the path control requirements of the mission are stringent, it is necessary to use sensors with a low threshold level, i.e. high sensitivity. If deviations are large and accuracy requirements are still stringent, it is necessary to use sensors with a larger dynamic range and low scale factor error. In all, the analysis of disturbances has a number of implications to the later choice of sensors.

In order to illustrate the possible connection between deviations and the choice of a correction concept, one may anticipate some comments below about correction possibilities. An important one is based upon the expansion of true position about nominal position in a Taylor series in the deviations and basing the correction calculations on only the first order terms in small deviations. Clearly, if the deviations become large the second and higher order terms of the expansion become much more important and it may introduce serious errors if the correction is made only on the basis of the linearized form. Large deviations also assume particular importance for the case of low thrust vehicles, since it may become impossible to choose any path control concept which permits their correction within the thrust limitations of the propulsion system. The transition from existence to non-existence of a correction concept is obviously a rigorous one and nicely illustrates that the magnitude of deviations indeed can have an important effect. Actually, one might expect a whole series of gross changes in the required concept as the limits of thrust capability are approached.

Let us turn now to the question of sensors. It is perhaps unfortunate that so much stress has been placed in the literature on the methods of sensing position and velocity of astronautical vehicles. This has resulted in a tendency to regard this sensing operation alone as "the" problem of space navigation, whereas it is actually just one facet whose importance should be neither minimized nor over-magnified. Considerable ingenuity has gone into the question of establishing a location in space. Broadly, proposed methods are of four major classes: sightings on distant bodies (effectively point sources), measurements on ambient fields of the vehicle, line of sight observations on nearby bodies, and observations on the vehicle from other sites.

Stars may be used as point sources for establishing orientation, but position inference by similar observations involves distant planets. The process is essentially one of triangulation, using one's knowledge about the geometric location of the planets. The literature contains both the basic equations to be solved and expressions for the position errors resulting from angular measurement errors and the error in the astronomical unit of distance. Velocity can be inferred most practically by differencing successive position measurements, but it also has been suggested that the Brewster effect or doppler shifts might be used.

The simplest examples of position and velocity determination by field sensing occur in the neighborhood of planets where an atmosphere and magnetic field might exist. Even the gravitational field is of potential use to infer distance from a nearby body if sufficiently precise measurement of the gravity gradient can be made aboard the vehicle, e.g. by pairs of very sensitive accelerometers. Still another method is to determine distance from a radiating body (typically the sun) by absolute measurement of received radiation at the vehicle.

Line of sight observations on a close body offer a number of possibilities, such as triangulation by either angle or absolute distance (by radio) measurements using three points or sources whose geographic locations on the body are known. Observations on the periphery of the presented disc, in the manner of horizon scanning often discussed for attitude determination, offer the possibility of determining radial distance provided the diameter of the body is known and one can accept such systematic errors as those from oblateness and refraction.

The final class, observations on the vehicle from other locations, is a particularly important one. A large number of possible methods is represented, including optical and radio methods. More than one external location may be involved, perhaps the planet of origin or des-

tination or on another space vehicle. Cooperative arrangements using beacons or transponders on the vehicle would be included here. It should be recognized that if these sensing methods are to be included in the control loop, an additional communication of command information to the vehicle is required. This, among other operational considerations, would suggest that such sensing is of value principally in low frequency applications such as occasional impulsive course correction. But frequent position and velocity information generally is necessary only to initiate maneuvers near a planet, so periodic position checks supplemented by interim prediction may suffice in any case. One advantage of this class is that complete geographic location including not only radial distance but also the equivalent of a latitude and longitude can be obtained, information which cannot be extracted from certain of the other methods.

One other point about sensing also must be remembered, namely that in path control the sensing plays two roles. It not only determines position and/or velocity, but also establishes the reference frame with respect to which the path control operations should be conducted. For the latter, there is a possibility of integrating path and attitude control sensors as shown in Figure 1.

The sensors discussed above form the portion of the path control problem traditionally called guidance. In the same way, the actuators together with some of the control loops comprise the traditional flight control area. Actuator characteristics are of obvious importance in making any error studies of a complete system, but also may have an effect upon the choice of a control concept. For example, the relative accuracy with which a thrust device can be pointed and turned off or on could determine which of these two operations was stressed in this concept. There are two basic forms which actuation takes in path control: one is to change the direction of the line of thrust so as to change the direction in which the center of mass of the system moves, and the other is to initiate and terminate thrust. The direction of the thrust in space normally is changed by changing the orientation of the entire vehicle frame: this, in turn, is accomplished by changing the direction of the line of thrust relative to vehicle fixed axes so that it no longer passes through the vehicle center of mass. The physical mechanism for accomplishing the change of direction relative to the vehicle can be either a gimballed engine or fixed side thrust jets. The fact that the rotational mode of the vehicle arises in this problem implies that there are really two control loops to consider. One of these is the relatively high frequency stabilization of the vehicle attitude during powered flight, so that the line of thrust is kept properly oriented in inertial space. The other is the lower frequency guidance mode which pertains to the gross motion of the center of mass. Both of these are proper ingredients of the path control problem.

The path control concepts deserve considerably more attention than can be given them in this brief descriptive review. It is convenient to visualize three varieties: concepts for the establishment of a prescribed initial state (e.g. position and velocity) where the path includes a main boost period, those for the change of state by means of impulsive or quasi-impulsive maneuvers; those applicable to low thrust correction systems. In the first, the concept for the correction of deviations from a nominal path may be any of those applicable to a ballistic missile, which is the case that probably has received the most extensive past study. In the open literature, the concept generally described is the so-called "delta method". The position and velocity deviations from the nominal trajectory are monitored during main boost. If cut-off should occur at any instant, the corresponding deviators at the terminal point (i.e. a point following a coasting period at which the mission ends or some further maneuver is initiated) are given a Taylor series in the instantaneous position and velocity deviations, often approximated closely by just the linear terms in the series. Predicted deviations at the terminal point can be used in at least two ways. For instance, the steering can be modified (e.g. by changing the line of thrust as some linear function of observed deviations) and the cut-off time can be advanced or retarded so as to make at least some of the terminal deviations equal to zero. Alternately, the original steering and cut-off program are unchanged while the observed cut-off deviations are used to compute a new course of action for the maneuver to be performed at the terminal point. In general, the actual concepts for a specific mission involve a combination of these.

At the same time, high frequency stability of rotational modes is necessary during boost, so it is necessary to include an additional control loop which assures that the vehicle attitude is stable relative to the value prescribed by the course of action above. It is well known that this is an exceedingly difficult task because of vehicle flexibility and change in its parameters as fuel is used, but obviously it is a problem which has been solved in at least some cases.

The change of elements by impulsive maneuvers is an elementary topic in astrodynamics and it would appear that the path control concept is quite straightforward. This is largely true: one need merely choose an initial vector velocity increment which makes the desired changes. The first difficulty is that only three velocity components are subject to control, whereas it may be desired to change more than three elements. The second is that any velocity component generally changes elements other than those desired. These considerations mean that more than one velocity impulse generally is needed, and that several may be desirable. Errors in the physical realization of any impulse affect the state of the system at the time the next impulse normally would be delivered. Thus the next impulse plays not only its nominal role (a purely astrodynamical question) but

must also correct the errors resulting from previous maneuvers. This, then, is the problem of the maneuver concept from the path control point of view.

Very little has been said in the literature about this propagation of errors through a general maneuver chain, and even less about correction philosophies. The case of low thrust error correction also is a topic which has received only a little attention.

Attitude Control

I have dwelt on the attitude control problem area in considerable detail elsewhere [34], so that only a brief review is appropriate here. I have found it convenient to divide the topic into the following areas: fundamentals of foundation problems, disturbing torques, attitude sensors, actuators, and control system concepts for controller synthesis. In many ways these closely parallel the major areas into which path control has been divided above.

The foundation questions involve primarily the controlled element dynamics, including the choice of an operationally suitable reference system. They also encompass performance measures and synthesis criteria. The development of the attitude dynamics in the form of differential equations of attitude motion is not a trivial matter since the vehicle ordinarily consists of a collection of parts that can move relative to one another. It should be emphasized, though, that even if the vehicle were a single rigid body the controlled element dynamics could not be idealized simply as a transfer function—the full differential equations generally must be used. Two major varieties of attitude reference are common: the "body-centered" system in which one axis is directed approximately toward the center of a nearby body and another axis is in the forward direction of motion about this body; and the inertial system in which the orientation of the reference frame is fixed in inertial space. Within the former class a variety of detailed possibilities exist. The remaining foundation problems of performance representation and synthesis criteria present problems which have not yet been treated adequately in connection with attitude control systems.

Disturbing torques on satellite or space vehicle arise from many physical effects. For a vehicle containing moving parts, especially crew members, this can be expected to be the major source of disturbance. Other torques arise from non-uniform rotation of the orientation reference frame, gravity gradient effects, electromagnetic interaction with ambient fields, aerodynamic drag in the fringes of an atmosphere, bombardment by particles or radiation (especially solar radiation). The analysis of the disturbing torques has been discussed in the previous reference and it is wished to emphasize only one point here. This is the fact that the torque structure on the vehicle is extremely sensitive to the configuration design and to such operational details as the regions of space through which the vehicle passes. One would expect the dependence on mission but migh not be aware of the importance of vehicle design detail

Considerable attention has been given to the analysi and development of sensors suitable for determinin attitude, particularly attitude relative to a close bod when the attitude reference is to be body-centere However, one need not consider all of the resultin methods since one of them, the horizon scanner, seem to suffice to determine the vertical for most body centered applications. Even for this one type of metho various mechanizations are possible, but the propert they have in common is that they somehow determin the center of the apparent disc of the nearby body as i is presented to an observer or instrument in the vehicl Complete attitude determination includes yaw as we as vertical, so the horizon scanner or equivalent devic must be supplemented by a yaw sensor, one type o which is a gyrocompass. Rate sensing sometimes is important and can be done by rate gyros or celestial rat sensors, the latter, based on the apparent drift of th star field, appearing particularly attractive from thei overall design characteristics. Alternatively, one ca dispense with rate sensing and attain the necessar control system damping by differentiation of attitud signals within the controller. If the desired attitud reference is inertial rather than body centered, th most attractive sensors appear to be star trackers possibly supplemented by an inertial platform.

In general, three kinds of control torques need to b applied. First, one must apply relatively high torqu levels to stop any initial tumbling of the vehicle. Second one must apply torques of somewhat less magnitud and generally of oscillatory character to counteract th disturbances during the operational life of the vehicl These two cases would suffice if there were no constan torque components acting over a long period of time but since there normally are such torques the tota angular momentum storage capacity of any control device would eventually become saturated. This means in effect, that a third type of torque need be applied— "trimming" torque to prevent angular momentum saturation of the primary actuator by providing a path for excess angular momentum to escape from the vehicl to inertial space.

The problem of controller synthesis is the leas studied of the major problems of attitude control, primarily because of the complexity of the controlle element dynamics and the fact that it is not yet know with what degree of approximation one can represen the vehicle by a conventional transfer function. However, some synthesis techniques are in the process o development. Analog computer studies of control system response using various controller functions hav been made by a number of organizations. From exten results it appears that controller design is not a formidable a problem as the present author had originally envisioned, although it is not a trivial one. On can design the system so that its accuracy is essentiall

that of the sensors provided enough control power can be made available. However, in practical systems, it often is preferable to relax the accuracy requirements in favor of the considerable saving in power consumption that results from the broader accuracy tolerances. Thus the most economical system from the point of view of power consumption (and space vehicles do tend to be power starved), is one which permits a slow limit cycle oscillation through several degrees. On the other hand, accuracies of a fraction of a second of arc seem to be attainable if one is willing to pay the price in power.

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Optimum Two Impulse Ascents to Circular Orbits

J. P. Carstens* and T. N. Edelbaum*

Abstract

The problem of launching an earth satellite into a circular orbit of arbitrary radius and angle of inclination with respect to the launch point has recently been posed by Wolfe and DeBra [1]. It is the intention of this note to indicate that, contrary to the statements in [1], target orbit interception at the apogee of the transfer ellipse is not generally optimum, and leads to non-minimum values of required total characteristic velocity.

Discussion

Releasing the restriction that interception of the target orbit must occur at the apogee of the transfer ellipse yields the results shown in Fig. 1, which is compared to similar data presented in Fig. 5 of [1]. Figure 1 was determined numerically on a high speed digital computer by allowing both θ and B, to vary for any given combination of r and i_{min}. The assumption of orbit injection at the apogee of the transfer ellipse is evidently justified only for very low values of i_{min} (exactly so for $i_{min} = 0$), and for all values of i_{min} with large r.

The values of θ for the optimum maneuver are compared to the values required for injection into orbit at apogee in Fig. 2. θ_{opt} drops below 90 deg for small values of r even

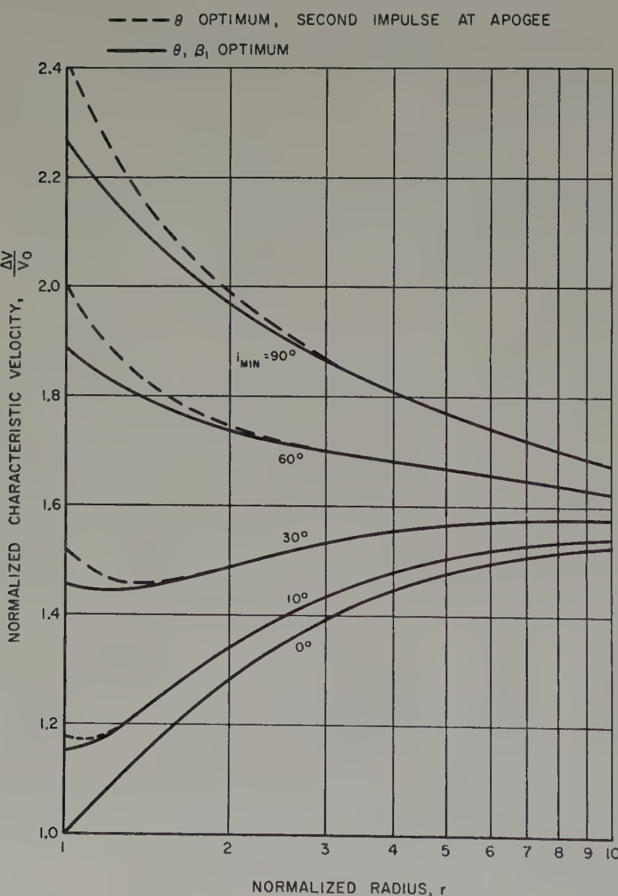

Fig. 1

* Research Laboratories, United Aircraft Corporation, East Hartford, Conn.

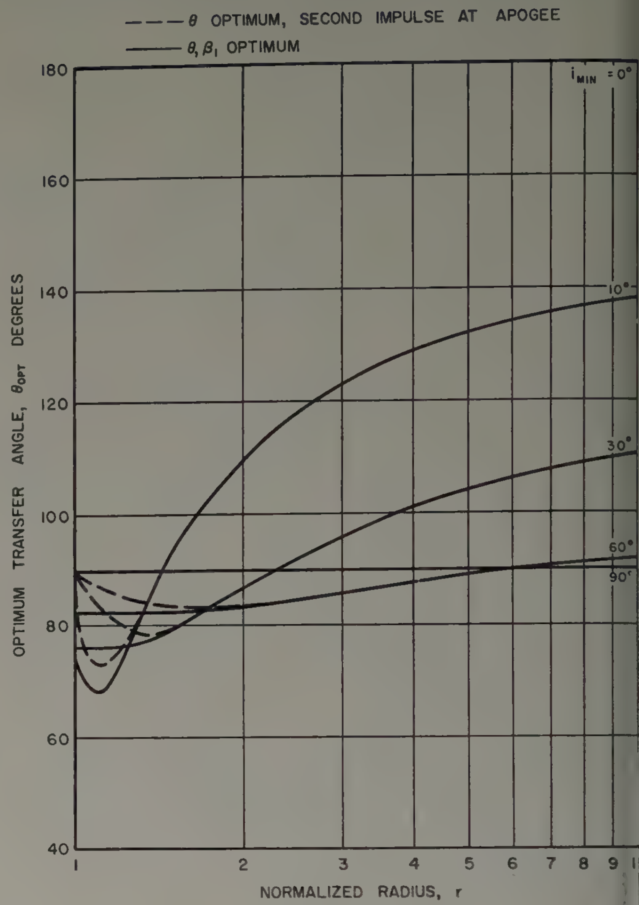

Fig. 2

for apogee injection, in variance with Fig. 4 of [1]. Optimur values of a second trajectory variable, β_2, the angle betwee the velocity vector in the transfer ellipse and the radiu vector at the point of interception with the target orbit, ar presented in Fig. 3. $\beta_2 < 90$ deg corresponds to the satellit vehicle being lofted to a greater altitude than the targ orbit, the second impulse being given at the descendin node. For injection at apogee, $\beta_2 = 90$ deg.

The maximum difference between the two ascents evidentl occurs at $r = 1.0$ and $i_{min} = 90$ deg. For this case a saving in normalized characteristic velocity of 0.152 is availabl with the optimum maneuver over the trajectory of [1]. Fo low altitude earth orbits this would correspond to a cha acteristic velocity saving of about 4000 fps and would allo typical increases in payload with chemical rockets of fro 40 to 55% [2].

A more complete discussion of this problem and the opt mum transfer ellipses may be found in [3].

Nomenclature

- i_{min} Minimum possible angle between plan containin launch point and center of earth and plane of targ orbit.
- r Nondimensional radius, normalized with respect t radius of launch point.
- V_0 Circular velocity at launch point
- ΔV Total change in velocity of first and second impuls
- β_1 Angle between velocity vector and radius vector launch point
- β_2 Angle between velocity vector and radius vector interception of target orbit
- θ Angle between radius vector of launch point and radi vector at point of interception of the target orbit.

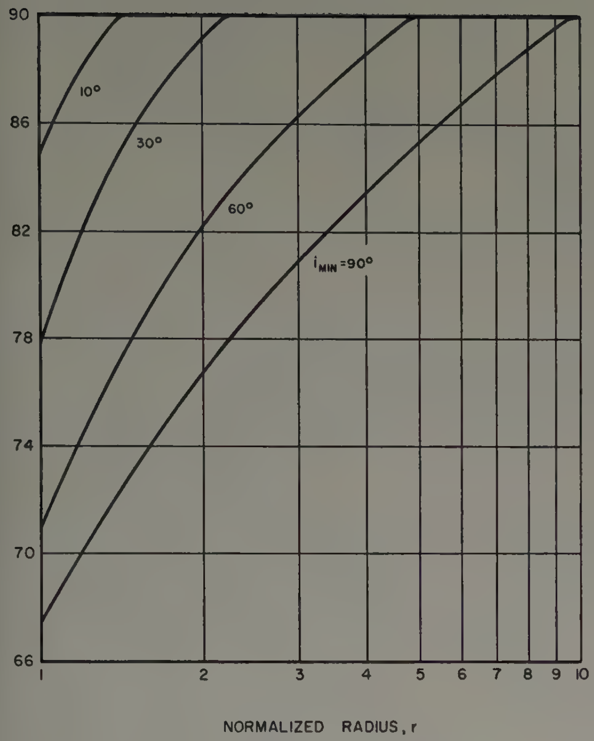

FIG. 3

References

[1] D. DEBRA AND J. F. WOLFE, "Two Maneuver Ascents to Circular Orbits," *The Journal of Astronautical Sciences*, Vol. VIII, No. 2, Summer, 1960,

[2] EDELBAUM, T. N., "Preliminary Comparison of Air and Ground Launching of Satellite Rendezvous Vehicles." UAC Research Laboratories Report UAR-0719, June 6, 1960.

[3] CARSTENS, J. P. AND EDELBAUM, T. N. "Optimum Maneuvers for Launching Satellites into Circular Orbits of Arbitrary Radius and Inclination. AROS Preprint 1450-60. December, 1960.

Comment by D. DeBra* on the Note by Carstens and Edelbaum: *Optimum Two Impulse Ascents to Circular Orbits*

When Mr. Edelbaum sent us a copy of this note he pointed out that the case where i_{min} is equal to 90 degrees and r is equal to 1.0 can be easily verified as a counter example to Fig. 4 and Fig. 5 of [1]. In this case i must be constant and the minimum characteristic velocity occurs for the minimum semi-major axis (energy) of the transfer orbit. The maximum error in our results also occurs for this case as pointed out in this note by Carstens and Edelbaum.

Our analysis of the minimum characteristic transfer orbit was checked for a value of r larger than 3.0. It can be seen from Fig. 1 why this check did not help to discover the error since the two results rapidly converge above this value of the radius. All of which goes to show that "the usual methods for finding extrema" must always be used with caution and checked at many points to avoid missing errors of this type. Our analysis was performed with the additional constraint of second maneuver at apoapsis and therefore led to non-optimum transfer orbit as reported here.

It is surprising that as r became large the two results should apparently coincide so that the asymptotes of Fig. 4 of [1] are still correct.

The results of Carstens and Edelbaum are interesting, and for small r, an important correction of [1].

Reference

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* Lockheed Aircraft Corporation, Sunnyvale, California.

Calendar of Events

1960

Nov. 21–26	Colloquium on Space Research—Argentine National Committee for Space Research and Argentine Interplanetary Society, Buenos Aires, Argentina.
Dec. 5–8	ARS Annual Meeting and Astronautical Exposition, Shoreham Hotel, Washington, D. C.
Dec. 27	Symposium on Lunar Exploration—Sponsored by American Astronautical Society. To be held as part of the 127th Annual Meeting of the American Association for the Advancement of Science, New York, N.Y.

1961

Jan. 16–18	7th Annual Meeting, American Astronautical Society, Sheraton Hotel, Dallas, Texas.
Mar. 17	AAS Special Symposium on Space Vehicles in Ionized Atmosphere, Washington, D. C.

NASA TO CO-SPONSOR LUNAR SYMPOSIUM

The Lunar Exploration Symposium to be held by the AAS as part of the 127th Annual Meeting of the American Association for the Advancement of Science will be co-sponsored by NASA. Ross Fleisig, Chairman of the Symposium, has announced the meetings will be held on December 27, 1960 in the Bowman Room of the Biltmore Hotel in New York City. In addition to morning and afternoon sessions, an evening panel discussion on Bio-Astronautics will be held. Chairman of the Panel will be Alfred M. Mayo, Assistant Director for Bio-Engineering in the NASA Office of Life Sciences.

The preliminary program for the Symposium is as follows:

December 27, 1960

SESSION I - LUNAR EXPLORATION, 9 A.M.

CHAIRMAN: Dr. Hugh L. Dryden, Deputy Administrator, National Aeronautics and Space Administration

PAPER I

Scientific Objective of Lunar Exploration, ROBERT JASTROW, Goddard Space Flight Center, NASA

PAPER II

Distribution of Interplanetary Dust in Cislunar Space, S. F. SINGER, Department of Physics, University of Maryland

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PAPER III

Comparison of Special Pertubation Methods in Celestial Mechanics with Special Application to Lunar Orbits, SAM PINES, Applied Mathematics Section, Republic Aviation Corporation

PAPER IV

Radiation Shielding for Lunar Spacecraft, T. G. BARNES, A. L. BARAZOTTI and E. M. FINKELMAN, Grumman Aircraft Engineering Corporation

SESSION II - LUNAR SPACE SPACE SYSTEMS, 2 P.M.

CHAIRMAN: Robert Young, Executive Vice President; ACF Electronics Division, ACF Industries, Inc.

PAPER I

A Family of Radio Isotope-Fueled Auxiliary Power Supplies for Lunar Exploration, JUSTIN L. BLOOM, Martin Company

PAPER II

Extending the Range of Radar Beacon Tracking for Lunar Probes, N. S. GREENBERG, ACF Electronics Division, ACF Industries, Inc.

PAPER III

Horizon Trackers for Lunar Guidance and Control Systems, K. H. KUHN, Sperry Gyroscope

PAPER IV

Lunar Soft Landing Guidance Sensors, GORDON BURTON and ARTHUR BARABUSH, Raytheon Missile System Division, Raytheon

PANEL DISCUSSION

Is There a Need for a Manned Space Laboratory?–December 27, 1960, 8 P.M.

MODERATOR—A. M. Mayo, Assistant Director for Bio-Engineering, NASA

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SEVENTH ANNUAL MEETING TAKING SHAPE

PAPERS GIVE PROMISE OF TOP-FLIGHT MEETING

The call for papers for the Sevent Annual Meeting to be held at Dalla January 16–18 went out on July 7, an the excellent response indicates AA can expect sessions of unusual interest

LEADING AUTHORITIES SELECTED AS SESSION CHAIRMEN

The Southwest Section, host for th meeting, is striving to make the Sevent Annual Meeting a session of exceptiona benefit to the aerospace industry. As a indication of this, a number of the na tion's leading authorities in the field c astronautics have accepted posts as ses sion chairmen to assist in obtaining to quality papers covering virtually ever phase of space technology. The AAS i proud to announce these men as chairme of the technical sessions:

- Dr. Samuel Herrick, University c California, Astrodynamics Session
- Dr. Otto Struve, National Radio As tronomy Observatory, Cosmolog Session
- Dr. S. F. Singer, University of Mary land, Solar System Physics Session
- Dr. Homer E. Newell, National Aer onautics and Space Administratio Geophysical Exploration Session
- Dr. Thomas Gold, Cornell Universit Lunar and Planetary Exploratio Session
- Mr. George H. Stoner, Boeing Ai plane Company, Space System Applications Session
- Mr. Henri Busignies, Internationa Telephone and Telegraph Corpora tion, Space Communications Sessio
- Brig. Gen. Don Flickinger, Air Re search and Development Comman Bio-Astronautics Session
- Mr. H. Julian Allen, NASA Ames Re search Center, Re-Entry and Re covery Session

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List at least four references familiar with your activities. Complete addresses of non-members must be s plied. However, references need not be A.A.S. members.

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Date, From - To (year only)	Name of Affiliation and Description of Duties

Format of Technical Papers for AAS Journal

The Editors will appreciate the cooperation of authors in using the following directions for the preparation of manuscripts. These directions have been compiled with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts

Papers should be submitted in original typewriting (if possible) on one side only of white paper sheets, and should be double or triple spaced with wide margins. However, good quality reproduced copies (e.g. multilith) are acceptable. An additional copy of the paper will facilitate review.

Company Reports

The paper should not be merely a company report. If such a report is to be used as the basis for the paper, appropriate changes should be made in the title page. Lists of figures, tables of contents, and distribution lists should all be deleted.

Titles

The title should be brief, but express adequately the subject of the paper. A footnote reference to the title should indicate any meeting at which the paper has been presented. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Abstracts

An abstract should be provided, preceding the introduction, covering contents of the paper. It should not exceed 200 words.

Headings

The paper can be divided into principal sections as appropriate. Headings or paragraphs are not numbered.

Illustrations

Drawings should be made with black India ink on white paper or tracing cloth, and should be at least double the desired size of the cut. Each figure number should be marked with soft pencil in the margin or on the back of the drawing. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Reproducible glossy photographs are acceptable. However, drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be typewritten on a separate sheet, properly identified.

Security Clearance

Authors are responsible for the security clearance by an appropriate agency of the material contained in the papers.

Mathematical Work

As far as possible, formulas should be typewritten. Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Each such symbol should be identified unambiguously the first time it appears. The distinction between capital and lower-case letters should be clearly shown. Avoid confusion between zero (0) and the letter O; between the numeral (1), the letter l, and the prime ($'$); between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts, and exponents in exponents should be clearly indicated.

Greek Alphabet

Α	α	alpha	(a)	Ν	ν	nu	(n)
Β	β	beta	(b)	Ξ	ξ	xi	(x)
Γ	γ	gamma	(g)	Ο	o	omicron	(ŏ)
Δ	δ	delta	(d)	Π	π	pi	(p)
Ε	ϵ	epsilon	(ĕ)	Ρ	ρ	rho	(r)
Ζ	ζ	zeta	(z)	Σ	σ ς	sigma	(s)
Η	η	eta	(ē)	Τ	τ	tau	(t)
Θ	θ	theta	(th)	Υ	υ	upsilon	(u)
Ι	ι	iota	(i)	Φ	$\phi\varphi$	phi	(ph)
Κ	κ	kappa	(k)	Χ	χ	chi	(ch)
Λ	λ	lambda	(l)	Ψ	ψ	psi	(ps)
Μ	μ	mu	(m)	Ω	ω	omega	(ō)

ϵ (epsilon) = strain

σ, ς (sigma) = stress

τ (tau) = shear stress

μ (mu) = micro

μ (mu) = Poisson's Ratio

Complicated exponents and subscripts should be avoided when possible to represent by a special symbol.

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus:

$$\frac{\cos(\pi x/2b)}{\cos(\pi\alpha/2b)}$$

is the preferred usage.

The intended grouping of handwritten formulas can be made clear by slight variations in spacing, but this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus

$(a + bx)\cos t$ is preferable to $\cos t\,(a + bx)$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of braces, brackets, and parentheses (which should be used in this order). Thus:

$$\{[a + (b + cx)^n]\cos ky\}^2$$

is required rather than $((a + (b + cx)^n)\cos ky)^2$.

Equations are numbered and referred to in text as (15).

Bibliography

References should be grouped together in a bibliography at the end of the manuscript. References to the bibliography should be made by numerals between square brackets [4].

The following examples show the approved arrangements:

for books—[1] HUNSAKER, J. C. and RIGHTMIRE, B. S., *Engineering Applications of Fluid Mechanics*, McGraw-Hill Book Co., New York, 1st ed., 1947, p. 397.

for periodicals—[2] Singer, S. F., "Artificial Modification of the Earth's Radiation Belt," *J. Astronaut. Sci.*, *6* (1959), 1–10.